

Spectral function in QED₃

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Abstract

We discuss the structure of the dressed fermion propagator in unquenched QED₃ based on spectral function of photon. In this approximation infrared divergences that appeared in quenched case turns out to be soft. The dimension full coupling constant naturally appears as an infrared mass scale in this case. We find the reliable results for the effects of vacuum polarization for the dressed fermion propagator. The lowest order fermion spectral function has logarithmically divergent Coulomb energy as well as self-energy, which plays the role of confinement and dynamical mass generation. In our model finiteness condition of vacuum expectation value is equivalent to choose the scale of physical mass which is expected in the $1/N$ approximation.

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I. INTRODUCTION

About 25 years ago it was pointed out that high temperature limit of the field theory is described by the same theory with less-dimension and it suffers from severe infrared divergences with dimensionful coupling constant[1]. Before these argument infrared divergences

associated with massless particle such as photon and graviton has been discussed to determine the infrared structure of one particle state[2]. These are known as cut structure or infrared behaviour of the propagator near mass shell. In this kind of works renormalization group analysis or method of spectral function had been applied[2]. However in the analysis of three dimensional theory these approaches have not been used. The main reason may be in the dimensionfull coupling constant and the super renormalizability of the model. But in[2,3,4] infrared divergence near the mass shell has been given in a model independent way. Of course the topological mass soften infrared divergence but it is limited to parity violating case and we concentrate ourselves to parity conserving case[5]. At the same time (2+1) dimensional QED with massless fermion has been shown its dynamical chiral symmetry breaking by the analysis of Dyson-Schwinger (D-S) equation and Lattice simulation [6,7,8,9,10,11,12] and the massless pair instability has been shown by solving Behte-Salpeter (B-S) equation with $1/N$ approximation[13]. In condensed matter physics (2+1)-dimensional fermionic system shows super-conductivity phase. It has been pointed out that the d-wave superconductor-insulator transition at $T = 0$ is analougas to the dynamical mass generation in QED₃. And QED₃ is referred to as an effective theory of phase transition for its pairing instability of massless fermion[14,15]. In the previous work we studied fermion propagator based on the mass singularity in QED₃ in quenched approximation with finite bare photon mass in spectral representation[3]. After exponentiation one photon matrix element we have the explicit form of the propagator with confining properties, dynamical mass generation as we got in the D-S equation. The purpose of this paper is to improve quenched approximation by taking into account of vacuum polarization. Other approaches has shown that the screening effects soften the infrared behaviour [1,4,6]. Especially in [1] order $e^4 \ln(e^2)$ self-energy of massless fermion was improved by spectral function of photon with vacuum polarization of massless fermion. In this work we use the same spectral function of dressed photon propagator and get the non-perturbative effects by integrating the quenched spectral function of fermion with bare photon mass. As a result linear infrared divergences turned out to be logarithmic one, and $\ln(\mu |x|)$ is converted to $\ln(e^2 |x|)$ in the spectral function. Thus we can remove an infrared cut-off from the dressed fermion propagator away from the threshold, where the coupling constant e^2 plays the role of photon mass. In comparison our results with the analysis by unquenched Dyson-Schwinger equation, our solution is consistent with the latter excepts for wave renormalization $Z_2^{-1} = 0$. In section II spectral representation of the

fermion propagator are defined and we show the way to determine it based on LSZ reduction formula and low-energy theorem. In section III we evaluate the full propagator for quenched case with photon mass as an infrared cut-off and improve it by the photon spectral function for unquenched case. In Section IV is devoted to the analysis in momentum space of these solutions and the comparison with Dyson-Schwinger analysis[9,11,12].

II. CALCULATING THE SPECTRALY WEIGHTED PROPAGATOR

A. Definition of the spectral function

In this section we show how to evaluate the fermion propagator non perturbatively by the spectral representation which preserves unitarity and analyticity[2,3,4]. The spectral function of the fermion is defined

$$S_F(s') = P \int ds \frac{\gamma \cdot p \rho_1(s) + \rho_2(s)}{s' - s} + i\pi(\gamma \cdot p \rho_1(s') + \rho_2(s')), \quad (1)$$

$$\begin{aligned} \rho(p) &= \frac{1}{\pi} \text{Im} S_F(p) = \gamma \cdot p \rho_1(p) + \rho_2(p) \\ &= (2\pi)^2 \sum_n \delta(p - p_n) \langle 0 | \psi(x) | n \rangle \langle n | \bar{\psi}(0) | 0 \rangle. \end{aligned} \quad (2)$$

In the quenched approximation the state $|n\rangle$ stands for a fermion and arbitrary numbers of photons,

$$|n\rangle = |r; k_1, \dots, k_n\rangle, r^2 = m^2, \quad (3)$$

we have the solution for the spectral function $\rho(p)$ which is written symbolically

$$\begin{aligned} \rho(p) &= \int \frac{m d^2 r}{r^0} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int \frac{d^3 k}{(2\pi)^2} \theta(k^0) \delta(k^2) \sum_{\epsilon} \right)_n \delta(p - r - \sum_{i=1}^n k_i) \\ &\times \langle \Omega | \psi(x) | r; k_1, \dots, k_n \rangle \langle r; k_1, \dots, k_n | \bar{\psi}(0) | \Omega \rangle. \end{aligned} \quad (4)$$

Here the notations

$$(f(k))_0 = 1, (f(k))_n = \prod_{i=1}^n f(k_i)$$

have been introduced to show the phase space of each photons. To evaluate the contribution of soft photons, first we consider the situation when only the n th photon is soft. We define the matrix element

$$T_n = \langle \Omega | \psi | r; k_1, \dots, k_n \rangle. \quad (5)$$

We consider T_n for $k_n^2 \neq 0$, continue off the photon mass shell by LSZ reduction formula:

$$\begin{aligned} T_n &= \epsilon_n^\mu T_{n\mu}, \\ \epsilon_n^\mu T_n^\mu &= \frac{i}{\sqrt{Z_3}} \int d^3y \exp(ik_n \cdot y) \square_y \langle \Omega | T\psi(x) \epsilon^\mu A_\mu(y) | r; k_1, \dots, k_{n-1} \rangle \\ &= -\frac{i}{\sqrt{Z_3}} \int d^3y \exp(ik_n \cdot y) \langle \Omega | T\psi(x) \epsilon^\mu j_\mu(y) | r; k_1, \dots, k_{n-1} \rangle, \end{aligned} \quad (6)$$

provided

$$\square_x T(\psi A_\mu(x)) = T\psi \square_x A_\mu(x) = T\psi(-j_\mu(x) + \frac{d-1}{d} \partial_\mu^x (\partial \cdot A(x))), \quad (7)$$

$$\partial \cdot A^{(+)}|phys\rangle = 0, \quad (8)$$

where d is a gauge fixing parameter. T_n satisfies Ward-Takahashi-identity

$$k_{n\mu} T_\mu^n(r; k_1, \dots, k_n) = e \exp(ik_n \cdot x) \langle \Omega | \psi(x) | r; k_1, \dots, k_{n-1} \rangle, \quad (9)$$

provided

$$\partial_\mu^y T(\psi(x) J_\mu(y)) = -e \psi(x) \delta(x - y). \quad (10)$$

By translational invariance

$$\psi(x) = \exp(-ip \cdot x) \psi(0) \exp(ip \cdot x), \quad (11)$$

we get the usual form

$$k_{n\mu} T_\mu^n(r; k_1, \dots, k_n) = e T_{n-1}(r; k_1, \dots, k_{n-1}), r^2 = m^2. \quad (12)$$

By the low-energy theorem the fermion pole term in T_μ^n is dominant for the infrared singularity in k_n^μ . Inclusion of regular terms and their contribution to Ward-identities are given for the scalar case[2]. Hereafter we consider the leading pole term for simplicity.

B. Approximation to the spectral function

One-photon matrix element T_1 which is given in [3,16]

$$\begin{aligned} T_1 &= \langle \Omega | \psi(x) | r; k \rangle = \left\langle in | T(\psi_{in}(x), ie \int d^3y \bar{\psi}_{in}(y) \gamma_\mu \psi_{in}(y) A_\mu^{\text{in}}(y)) | r; k \text{ in} \right\rangle \\ &= ie \int d^3y d^3z S_F(x - z) \gamma_\mu \delta^{(3)}(y - z) \exp(i(k \cdot y + r \cdot z)) \epsilon^\mu(k, \lambda) U(r, s) \\ &= -ie \frac{1}{(r + k) \cdot \gamma - m + i\epsilon} \gamma_\mu \epsilon^\mu(k, \lambda) \exp(i(r + k) \cdot x) U(r, s). \end{aligned} \quad (13)$$

Here $U(r, s)$ is a four component free particle spinor with positive energy. If we sum infinite numbers of photon in the final state as in (4), assuming pole dominance for k_n^μ we have a simplest solution to T_n in (12)

$$\begin{aligned} T_n|_{k_n^2=0} &= T_1 T_{n-1} \\ &= e^{\frac{\gamma \cdot \epsilon_n}{\gamma \cdot (r + k_n) - m}} T_{n-1}. \end{aligned} \quad (14)$$

From this relation we obtain the n -photon matrix element T_n as the direct products of T_1

$$\langle \Omega | \psi(x) | r; k_1, \dots, k_n \rangle \langle r; k_1, \dots, k_n, \bar{\psi}(0) | \Omega \rangle \rightarrow \prod_{j=1}^n T_1(k_j) \bar{T}_1(k_j). \quad (15)$$

In this way we have an approximate solution of (4) by exponentiation of the one-photon matrix element

$$\bar{\rho}(x) = -(i\gamma \cdot \partial + m) \int \frac{d^2 r}{(2\pi)^2 r^0} \exp(ir \cdot x) \exp(F), \quad (16)$$

$$\begin{aligned} F &= \sum_{\text{one photon}} \langle \Omega | \psi(x) | r; k \rangle \langle r; k | \bar{\psi}(0) | \Omega \rangle \\ &= \int \frac{d^3 k}{(2\pi)^2} \delta(k^2) \theta(k^0) \exp(ik \cdot x) \sum_{\lambda, s} T_1 \bar{T}_1. \end{aligned} \quad (17)$$

$$\begin{aligned} \sum_{\lambda, s} T_1 \bar{T}_1 &= \frac{(r+k) \cdot \gamma + m}{(r+k)^2 - m^2} \gamma^\mu \frac{(\gamma \cdot r + m)}{2m} \frac{(r+k) \cdot \gamma + m}{(r+k)^2 - m^2} \gamma^\nu \Pi_{\mu\nu} \\ &= -e^2 \left(\frac{\gamma \cdot r}{m} + 1 \right) \left[\frac{m^2}{(r \cdot k)^2} + \frac{1}{(r \cdot k)} + \frac{d-1}{k^2} \right], \\ r^2 &= m^2, k^2 = 0. \end{aligned} \quad (18)$$

Here $\Pi_{\mu\nu}$ is the polarization sum

$$\Pi_{\mu\nu} = \sum_{\lambda} \epsilon_\mu(k, \lambda) \epsilon_\nu(k, \lambda) = -g_{\mu\nu} - (d-1) \frac{k_\mu k_\nu}{k^2}. \quad (19)$$

And the free photon propagator has the form

$$D_0^{\mu\nu} = \frac{1}{k^2 + i\epsilon} \left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} + d \frac{k^\mu k^\nu}{k^2} \right]. \quad (20)$$

The fermion propagator is written explicitly in the following form in quenched case

$$\begin{aligned} S_F(x) &= -(i\gamma \cdot \partial + m) \frac{\exp(-m|x|)}{4\pi|x|} \\ &\times \exp(-e^2 \int \frac{d^3 k}{(2\pi)^2} \exp(ik \cdot x) \theta(k^0) \delta(k^2) \left[\frac{m^2}{(r \cdot k)^2} + \frac{1}{(r \cdot k)} + \frac{d-1}{k^2} \right]), \\ |x| &= \sqrt{-x^2}. \end{aligned} \quad (21)$$

Here $\delta(k^2)$ is read as the imaginary part of the photon propagator. For unquenched case we use the dressed photon with massless fermion loop with N flavours. Spectral functions for free and dressed photon are given by [1,3,4,17]

$$\begin{aligned}\rho^{(0)}(k) &= \delta(k^2 - \mu^2), \\ \rho^D(k) &= \text{Im } D_F(k) = \frac{c\sqrt{k^2}}{k^2(k^2 + c^2)}, c = \frac{e^2 N}{8}.\end{aligned}\quad (22)$$

In this case one photon matrix element is modified to

$$F = -e^2 \int \frac{d^3 k}{(2\pi)^2} \exp(ik \cdot x) [\text{Im } D_F(k) [\frac{m^2}{(r \cdot k)^2} + \frac{1}{(r \cdot k)} - \frac{1}{k^2}] + \frac{d}{k^4}]. \quad (23)$$

To evaluate the F we use the position space propagator in the next section. The spectral representation for photon propagator is written

$$\begin{aligned}iD_F(x) &= \frac{1}{\pi} \int \frac{d^3 k}{(2\pi)^3} \exp(ik \cdot x) \int_0^\infty dp^2 \frac{\rho^D(p)}{k^2 + p^2} \\ &= \int_0^\infty dp^2 \frac{\exp(-\sqrt{p^2} |x|)}{4\pi |x|} \rho^D(p^2),\end{aligned}\quad (24)$$

as a linear superposition of the photon with $\sqrt{p^2}$ mass.

III. APPROXIMATE SOLUTION IN POSITION SPACE

A. Quenched case

To evaluate the function F

$$\begin{aligned}F &= -e^2 \int \frac{d^3 k}{(2\pi)^2} \exp(ik \cdot x) \theta(k^0) \delta(k^2) [\frac{m^2}{(r \cdot k)^2} + \frac{1}{(r \cdot k)} + \frac{d-1}{k^2}] \\ &= F_1 + F_2 + F_L,\end{aligned}\quad (25)$$

it is helpful to use the exponential cut-off (infrared cut-off) [2,3,18]. In the appendices the way to evaluate F is given. By using the photon propagator with bare mass μ we obtain

$$F \simeq \frac{e^2(d-2)}{8\pi\mu} + \frac{\gamma e^2}{8\pi\sqrt{r^2}} + \frac{e^2}{8\pi\sqrt{r^2}} \ln(\mu |x|) - \frac{e^2}{8\pi} |x| \ln(\mu |x|) - \frac{e^2}{16\pi} |x| (d-3+2\gamma), \quad (26)$$

where γ is an Euler constant. In this case linear infrared divergence may be cancelled by higher order correction or away from threshold. At present we omit them here with constant

term. Linear term in $|x|$ is understood as the finite mass shift from the form of the propagator in position space (29) and $|x| \ln(\mu |x|)$ term is position dependent mass

$$m = \left| m_0 + \frac{e^2}{16\pi}(d-3+2\gamma) \right|, \quad (27)$$

$$m(x) = m + \frac{e^2}{8\pi} \ln(\mu |x|), r^2 = m^2, \quad (28)$$

which we will discuss in section IV. Here we see that the position space propagator is written as free one with physical mass m multiplied by quantum correction as

$$\frac{\exp(-m|x|)}{4\pi|x|} \exp(F) = \frac{\exp(-m|x|)}{4\pi|x|} (\mu|x|)^{D-C|x|},$$

$$D = \frac{e^2}{8\pi m}, C = \frac{e^2}{8\pi}. \quad (29)$$

From the above form we see that D acts to change the power of $|x|$ and plays the role of anomalous dimension of the propagator[3,18]. To search the stability of massless e^+e^- composite in the lattice simulation Colomb energy and self energy were considered[7]. Recently this problem has analysed in the B-S amplitude and shows the pairing instability which signals dynamical rearrangement of the vacuum[8]. In our case Coulomb energy for two electrons in two dimension is

$$\rho(x) = e\delta^{(2)}(x - x(t)), \rho(y) = e\delta^{(2)}(y),$$

$$-E_C = -\frac{e^2}{2} \int d^2x d^2y \rho(x) \frac{K_0(\mu|x|)}{\pi} \rho(y)$$

$$\rightarrow \frac{e^2}{2\pi} (\ln(\frac{\mu|x|}{2}) + \gamma) + O(\mu). \quad (30)$$

This is qualitatively the same with F_2

$$F_2 = -\frac{e^2}{8\pi} \text{Ei}(\mu|x|) \rightarrow \frac{e^2}{8\pi} (\gamma + \ln(\mu|x|)) + O(\mu).$$

If we compare these result with ours, $-F$ is a sum of self energy of electron and Coulomb energy of two electron e^-e^- and it is short ranged and positive at short distance $\mu|x| \leq 1$ due to the exponential cut-off. To see the difference between our approximation in three and four dimension, here we show the result in four dimension[2]. In four-dimension the photon propagator is

$$D_F^{(0)}(x)_+ = \frac{\mu K_1(\mu|x|)}{4\pi^2 i|x|} = \frac{1}{4\pi^2 i x^2} + O(\mu^2 \ln(\mu|x|)) + O(\mu^4). \quad (31)$$

We obtain by the parameter integral

$$F_1 = -ie^2 m^2 \int_0^\infty \frac{\alpha d\alpha}{4\pi^2 i (x + \alpha r)^2} \exp(-\alpha \mu r) = \frac{e^2}{8\pi^2} \ln(\mu^2 |x|^2), \quad (32)$$

$$F_2 = -e^2 \int_0^\infty \frac{d\alpha}{4\pi^2 (x + \alpha r)^2} = -\frac{e^2}{4\pi^2 \sqrt{r^2} |x|} = -\frac{e^2}{4\pi^2 m |x|}, \quad (33)$$

$$F_L = -i(d-1)e^2 \frac{\partial}{\partial \mu^2} \frac{\mu K_1(\mu |x|)}{4\pi^2 i |x|} = -\frac{(d-1)e^2}{16\pi^2} \ln(\mu^2 |x|^2) + O(\mu). \quad (34)$$

To evaluate F_L , $\delta(k^2)/k^2$ in (23) is replaced by $-\delta'(k^2)$ in the definition of F . The $-F_2$ is interpreted as the Coulomb energy of e^-e^- separated by $|x|$ divided by m . F_2 is finite in the infrared and does not contribute for the infrared singularity. Therefore the leading log correction leads the well-known form by Fourier transformation

$$S_F(p) = - \int d^4x \exp(-ip \cdot x) (i\gamma \cdot \partial + m) \frac{m K_1(m |x|)}{4\pi^2 i |x|} (\mu^2 |x|^2)^D$$

$$S_F(p) \simeq \frac{\gamma \cdot p + m}{m^2 (1 - p^2/m^2)^{1-D}}, D = \frac{\alpha(d-3)}{2\pi}, \alpha = \frac{e^2}{4\pi}, \quad (35)$$

near $p^2 = m^2$.

B. Unquenched case

Here we apply the spectral function of photon to evaluate the unquenched fermion proagator. We simply integrate the function $F(x, \mu)$ for quenched case which is given in (26), where μ is a photon mass. Spectral function of photon is given in (22) in the Landau gauge $d = 0$

$$\rho^D(\mu) = \frac{c}{\mu(\mu^2 + c^2)},$$

$$Z_3^{-1} = \int_0^\infty 2\rho(\mu) \mu d\mu = \pi. \quad (36)$$

An improved F is written as dispersion integral

$$\begin{aligned} \tilde{F} &= \int_0^\infty 2F(\mu) \rho^D(\mu) \mu d\mu \\ &= \frac{e^2}{8\pi c} (4) \ln\left(\frac{\mu}{c}\right) + \frac{\gamma e^2}{8m} + \frac{e^2}{8m} \ln(c|x|) \\ &\quad - \frac{e^2}{8} |x| \ln(c|x|) - \frac{e^2}{16} |x| (3 - 2\gamma), \end{aligned} \quad (37)$$

where linear divergent term is regularized by cut-off μ

$$\int_{\mu}^{\infty} \frac{2cd\mu}{\mu(\mu^2 + c^2)} = -\frac{2}{c} \ln\left(\frac{\mu}{c}\right). \quad (38)$$

In this way the linear infrared divergences turn out to be a logarithmic divergence in the first term and μ in the other logarithms is converted by c under the dispersion integral. Hereafter we neglect the $\ln(\mu/c)$ term in Euclid space. The fermion propagator with N flavours in position space is modified to

$$S_F(x) = -(i\gamma \cdot \partial + m)\bar{\rho}(x) \quad (39)$$

$$\begin{aligned} \bar{\rho}(x) &= \frac{\exp(-m|x|)}{4\pi|x|} \exp(\tilde{F}) \\ &= \frac{\exp(-|m_0 + B||x|)}{4\pi|x|} (c|x|)^{D-C|x|} \left(\frac{\mu}{c}\right)^{\beta} \exp\left(\frac{\gamma e^2}{8m}\right), \end{aligned} \quad (40)$$

$$B = \frac{c}{2N}(3 - 2\gamma), \beta = \frac{4}{N\pi}, C = \frac{c}{N}, D = \frac{c}{Nm}. \quad (41)$$

At large N the function damps slowly with fixed c , where mass changing effect is small. For small N the function damps fast and the short distant part is dominant for mass changing effect.

IV. IN MOMENTUM SPACE

Now we turn to the fermion propagator in momentum space. The momentum space propagator is given by Fourier transform

$$\begin{aligned} S_F(p) &= \int d^3x \exp(-ip \cdot x) S_F(x) \\ &= - \int d^3x \exp(-ip \cdot x) (i\gamma \cdot \partial + m) \frac{\exp(-m|x|)}{4\pi|x|} \exp(F(x)). \end{aligned} \quad (42)$$

where we have

$$\exp(F(x)) = A(c|x|)^{D-C|x|}, A = \exp\left(\frac{e^2\gamma}{8m}\right), D = \frac{c}{Nm}, C = \frac{c}{N}. \quad (43)$$

It is known that

$$\begin{aligned} & \int d^3x \exp(-ip \cdot x) \frac{\exp(-m|x|)}{4\pi|x|} (c|x|)^D \\ &= c^D \frac{\Gamma(D+1) \sin((D+1) \arctan(\sqrt{-p^2}/m))}{\sqrt{-p^2}(p^2 + m^2)^{(D+1)/2}} \end{aligned} \quad (44)$$

$$\begin{aligned} &= \frac{1}{p^2 + m^2} \text{ for } D = 0, \\ &= \frac{2m}{(p^2 + m^2)^2} \text{ for } D = 1. \end{aligned} \quad (45)$$

for Euclidean momentum $p^2 \leq 0$. $S_F(p)$ in Minkowski space is given in the Appendix B. To use the above formula for the case of mass generation $C \neq 0$ we use Laplace transform[3]

$$F(s) = \int_0^\infty d|x| \exp(-(s-m)|x|) (c|x|)^{-C|x|} (s \geq 0). \quad (46)$$

This function shifts the mass and we get the propagator

$$S_F(p) = (\gamma \cdot p + m) A c^D \Gamma(D+1) \int_0^\infty F(s) ds \frac{\sin((D+1) \arctan(\sqrt{-p^2}/(m-s)))}{\sqrt{-p^2}(p^2 + (m-s)^2)^{(D+1)/2}}. \quad (47)$$

At $D = 0$ and 1 we see

$$S_F(p) = (\gamma \cdot p + m) A \int_0^\infty F(s) ds \frac{1}{(p^2 + (m-s)^2)}, (D = 0). \quad (48)$$

$$S_F(p) = (\gamma \cdot p + m) A c \int_0^\infty F(s) ds \frac{2|m-s|}{(p^2 + (m-s)^2)^2}, (D = 1). \quad (49)$$

N dependence of $m\bar{\rho}(x)$ and its Fourier transform $m\rho(p)$ for $D = 1$ are shown in Fig.1 and Fig.2.

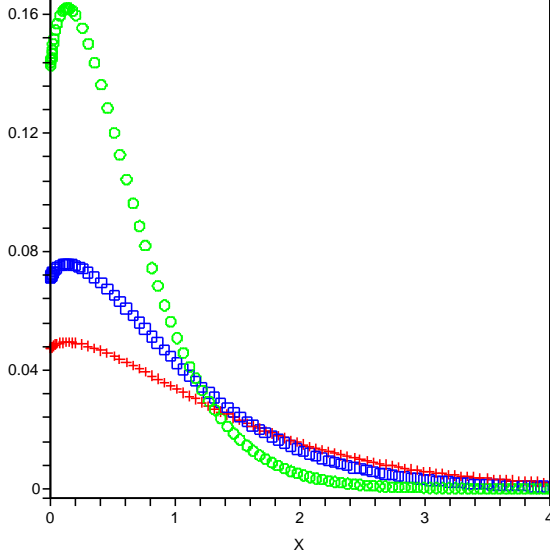


Fig.1 $m\bar{\rho}(x)$ for $N = 1(\bigcirc), 2(\square), 3(\times)$ in unit of c .

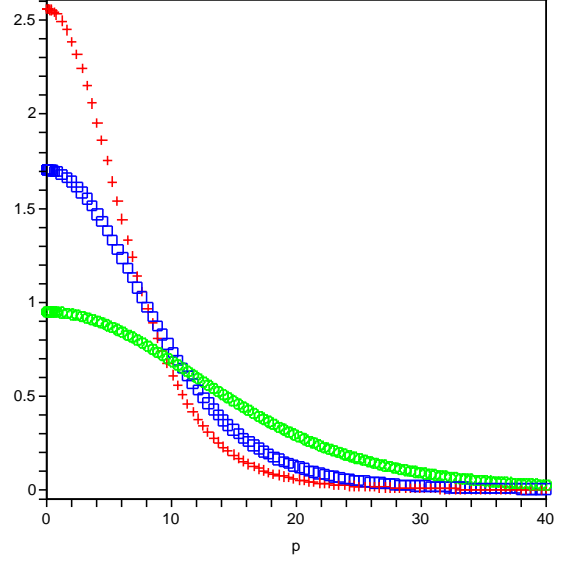


Fig.2 $m\rho(p)$ for $N = 1(\bigcirc), 2(\square), 3(\times)$ in unit of $1/c, p = .1n$.

V. RENORMALIZATION CONSTANT AND ORDER PARAMETER

In this section we consider the renormalization constant and bare mass in our model. It is easy to evaluate the renormalization constant and bare mass defined by the renormalization transformation

$$\begin{aligned} \psi_0 &= \sqrt{Z_2}\psi_r, \bar{\psi}_0 = \sqrt{Z_2}\bar{\psi}_r, \\ S_F^0 &= Z_2 S_F, \frac{Z_2^{-1}}{\gamma \cdot p - m_0} = S_F(p) = \int ds \frac{\gamma \cdot p \rho_1(s) + m \rho_2(s)}{p^2 - s}. \end{aligned} \quad (50)$$

$$\begin{aligned} Z_2^{-1} &= \int \rho_1(s) ds = \lim_{p \rightarrow \infty} \frac{1}{4} \text{tr}(\gamma \cdot p S_F(p)) \\ &= \Gamma(D+1) c^D \lim_{p \rightarrow \infty} \int_0^\infty F(s) ds \frac{\sqrt{-p^2} \sin((D+1) \arctan(\sqrt{-p^2}/(m-s)))}{(p^2 + (m-s)^2)^{(1+D)/2}} \\ &\rightarrow \sin\left(\frac{(D+1)\pi}{2}\right) c^D \lim_{p \rightarrow \infty} \frac{1}{\sqrt{-p^2}^D} = 0. \end{aligned} \quad (51)$$

$$m_0 Z_2^{-1} = m \int \rho_2(s) ds = \lim_{p \rightarrow \infty} \frac{1}{4} \text{tr}(p^2 S_F(p)) \rightarrow 0. \quad (52)$$

There is no pole and it shows the confinement for $D > 0$. Order parameter $\langle \bar{\psi}\psi \rangle$ is given as the integral of the scalar part of the propagator in momentum space

$$\langle \bar{\psi}\psi \rangle = -\text{Tr} S_F(x), \quad (53)$$

In position space we evaluate directly from (40)

$$\begin{aligned} \langle \bar{\psi}\psi \rangle &= 4 \lim_{x \rightarrow 0} m \bar{\rho}(x) \\ &= -\frac{4mc^D}{4\pi} \exp\left(\frac{\gamma e^2}{8m}\right) \lim_{x \rightarrow 0} x^{D-1}. \end{aligned} \quad (54)$$

From the above equation we see that $\langle \bar{\psi}\psi \rangle \neq 0$ and finite only if $D = c/Nm = 1$. In this case we have

$$\langle \bar{\psi}\psi \rangle = -\frac{c^2}{\pi N} \exp(\gamma) = -\frac{c^2}{\pi N} \cdot 1.7810.. \quad (55)$$

with the relation between physical mass and coupling constant

$$m = \frac{c}{N} = \frac{e^2}{8}. \quad (56)$$

The N dependence of $\langle \bar{\psi}\psi \rangle$ is weak at large N in comparison with Dyson-Schwinger equation where the order parameter vanishes fast with N [6,7,8,9,10,12]. It is not difficult to test the convergence for different values of D numerically in momentum space

$$\begin{aligned} \langle \bar{\psi}\psi \rangle &= -4m \exp\left(\frac{\gamma e^2}{8m}\right) \int_0^\infty \frac{p^2 d\sqrt{-p^2}}{2\pi^2} \frac{\Gamma(D+1)c^D}{2\sqrt{-p^2}} \\ &\times \int_0^\infty ds F(s) \frac{\sin((D+1) \arctan(\sqrt{-p^2}/(m-s)))}{\sqrt{((m-s)^2 + p^2)^{D+1}}}, \quad (D = c/Nm). \end{aligned} \quad (57)$$

At $D = 1$ we derive directly (55) provided

$$\int_0^\infty \frac{p^2 dp}{(p^2 + m^2)^2} = \frac{\pi}{4m}, \quad \int_0^\infty F(s) ds = 1. \quad (58)$$

We see vacuum expectation value is finite for $D = 1$. This condition is independent of the bare mass. However we cannot apriori determine the value D , since m is a physical mass. m and zero momentum mass $\Sigma(0)$ is not the same quantity in general but we assume they have the same order of magnitude. In our model $\Sigma(0)$ can be evaluated as the value of the inverse of the scalar part of the propagator at $p = 0$ with $m_0 = 0$. In numerical analysis of Dyson-Schwinger equation $\Sigma(0)$ damps fast with N and is seen to vanish at $N = 3$ [6,7,8,9,10,12]. In our approximation if we set m equals to the value B in (41) we obtain

$$D = \frac{2}{3 - 2\gamma} \simeq 1.08 \quad (59)$$

which is very close to 1. For large N , D remains $O(1)$ but C is $O(c/N)$ and mass generation is suppressed. For $D = 1$ we have a similar solution of the propagator at short distance which is known by the analysis of D-S. And we find the m and $\Sigma(0)$ are the same order. In the analysis of Gap equation, zero momentum mass $\Sigma(0)$ and a small critical number of flavours have been shown [6,7,8,9], in which same approximation was done because the vacuum polarization governs the photon propagator at low energy. The critical number of flavour N_c is a consequence of the approximation for the infrared dynamics as in quenched QED₄ where ultraviolet region is non trivial and is not easy to find numerically. For fixed value of the coupling we solved the coupled Dyson-Schwinger equation numerically and found that the $\Sigma(0)$ which is $O(e^2/4\pi)$ at $N = 1$, the same order of magnitude as we get in the quenched Landau gauge [11,12]. The value of $\langle \bar{\psi}\psi \rangle$ at $N = 1$ is the same order of magnitude 10^{-3} which is shown in [9]. In the case of vanishing bare mass $m_0 = 0$, Ward-Takahashi-identity for axial vector currents

$$\lim_{p \rightarrow q} (p - q)_\mu S_F(p) \Gamma_{5\mu}(p, q) S_F(q) = \{\gamma_5, S_F(p)\} \quad (60)$$

implies existence of a massless pole in the $\Gamma_{5\mu}(p, q) : F_\pi \chi(p - q)(p - q)_\mu / (p - q)^2$ where χ is a B-S amplitude for pseudoscalar bound state. This is related to the propagator

$$\begin{aligned} F_\pi \chi(p) &= 2m\rho(p)\gamma_5, \\ F_\pi \chi(x) &= 2m\bar{\rho}(x)\gamma_5. \end{aligned} \quad (61)$$

in our approximation. The scalar density $\bar{\psi}\psi(x)$ is understood as the density of electron. $\nabla \bar{\psi}\psi(x)$ gives the gradient energy of superfluid current. In our approximation or Dyson-Schwinger equation we can determine the propagator and the density of electron at the same time in the condensed phase.

VI. SUMMARY

We evaluate the fermion propagator in three dimensional QED with dressed photon by the method of spectral function. In the evaluation of lowest order matrix element for fermion spectral function we obtain finite mass shift, Coulomb energy and gauge invariant position dependent mass which has the same property in the the analysis of D-S equation. However there remains infrared divergences such as linear and logarithmic ones which were regularized by an infrared cut-off μ . Including vacuum polarization we find that infrared behaviour is

modified which is shown in (37)-(41) and we can avoid the infrared divergences away from threshold. This result is consistent with other analysis with dimensional regularization[4]. For $c/Nm = 1$ order parameter $\langle \bar{\psi}\psi \rangle$ is finite, where we have the familiar relation of the mass and coupling constant in $1/N$ approximation. In our analysis Coulomb energy at short distance is logarithmically divergent and plays the same role of mass singularity in four-dimension. This sets the renormalization constant $Z_2^{-1} = 0$ for arbitrary positive value of the coupling constant. If we assume the magnitude of physical mass m is $O(e^2)$ in the Landau gauge at $N = 1$, our results is consistent with numerical analysis of coupled Dyson-Schwinger equation excepts for wave renormalization [9,12]. We find the similar structure of the propagator in whole region except for N dependence. The advantage of our approximation is to get the super-fluid density $\bar{\psi}\psi(x)$ as well as B-S amplitude for pseudoscalar $\chi(x)$ by Ward-Takahashi-identity.

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IX. APPENDICES

A. Evaluation of the one-photon matrix element

In this section we evaluate the one-photon matrix element F in (23),(25). Following the parameter tric

$$\begin{aligned}\frac{1}{k \cdot r} &= i \lim_{\epsilon \rightarrow 0} \int_0^\infty d\alpha \exp(i\alpha(k + i\epsilon) \cdot r), \\ \frac{1}{(k \cdot r)^2} &= \lim_{\epsilon \rightarrow 0} \int_0^\infty \alpha d\alpha \exp(i\alpha(k + i\epsilon) \cdot r),\end{aligned}\tag{62}$$

we obtain the formulae to evaluate three terms in

$$F = -e^2 \left(\frac{\gamma \cdot r}{m} + 1 \right) \int \frac{d^3 k}{(2\pi)^2} \exp(ik \cdot x) \theta(k^0) \delta(k^2) \left[\frac{m^2}{(r \cdot k)^2} + \frac{1}{(r \cdot k)} + \frac{d-1}{k^2} \right].\tag{63}$$

First two terms are written explicitly by the photon ropagator $D_F(x)$

$$F_1 = - \int \frac{d^3k}{(2\pi)^3} \exp(ik \cdot x) D_F(k) \frac{1}{(k \cdot r)^2} = - \lim_{\mu \rightarrow 0} \int_0^\infty \alpha d\alpha D_F(x + \alpha r) \exp(-\mu \alpha r). \quad (64)$$

$$F_2 = - \int \frac{d^3k}{(2\pi)^3} \exp(ik \cdot x) D_F(k) \frac{1}{k \cdot r} = -i \lim_{\mu \rightarrow 0} \int_0^\infty d\alpha D_F(x + \alpha r) \exp(-\mu \alpha r). \quad (65)$$

Soft photon divergence corresponds to the large α region and μ is an infrared cut-off. It is simple to evaluate the last term in F by definition

$$F_L = \int \frac{d^3k}{i(2\pi)^3} \exp(ik \cdot x) D_F(k) \frac{1}{k^2}, \quad (66)$$

$$\frac{1}{4\pi^2} \int_0^\infty d\sqrt{k^2} \frac{\sin(\sqrt{k^2} |x|)}{\sqrt{k^2} |x| (k^2 + \mu^2)} = \frac{1 - \exp(-\mu |x|)}{8\pi\mu^2 |x|}. \quad (67)$$

We have

$$F = -ie^2 m^2 \int_0^\infty \alpha d\alpha D_F(x + \alpha r) - e^2 \int_0^\infty d\alpha D_F(x + \alpha r) + e^2 (d-1) \int \frac{d^3k}{i(2\pi)^3} \exp(ik \cdot x) D_F(k) \frac{1}{k^2}. \quad (68)$$

In quenced case the above formulae for the evaluation of three terms in F provided the position space propagator with bare mass

$$D_F^{(0)}(x)_+ = \int \frac{d^3k}{i(2\pi)^2} \delta(k^2 - \mu^2) \theta(k^0) \exp(ik \cdot x) \quad (69)$$

$$= \frac{1}{i(2\pi)^2} \int_0^\infty \frac{\pi \sqrt{k^2} d\sqrt{k^2} J_0(\sqrt{k^2} |x|)}{2\sqrt{k^2 + \mu^2}} = \frac{\exp(-\mu |x|)}{8\pi i |x|}. \quad (70)$$

$$F = -\frac{e^2}{8\pi} \left(\frac{\exp(-\mu |x|)}{\mu} - |x| \text{Ei}(\mu |x|) \right) - \frac{e^2}{8\pi\sqrt{r^2}} \text{Ei}(\mu |x|) + (d-1) \frac{e^2}{8\pi\mu^2 |x|} (1 - \exp(-\mu |x|)),$$

$$r^2 = m^2, \quad (71)$$

where

$$\text{Ei}(z) = \int_1^\infty \frac{\exp(-zt)}{t} dt, \quad (72)$$

$$\text{Ei}(\mu |x|) = -\gamma - \ln(\mu |x|) + (\mu |x|) + O(\mu^2). \quad (73)$$

For the leading order in μ we obtain

$$-e^2 F_1 = \frac{e^2}{8\pi} \left(-\frac{1}{\mu} + |x| (1 - \ln(\mu |x| - \gamma)) \right) + O(\mu), \quad (74)$$

$$-e^2 F_2 = \frac{e^2}{8\pi m} (\ln(\mu |x|) + \gamma) + O(\mu), \quad (75)$$

$$(d-1)e^2 F_L = \frac{e^2}{8\pi} \left(\frac{1}{\mu} - \frac{|x|}{2} \right) (d-1) + O(\mu). \quad (76)$$

Since we used the photon propagator with bare mass μ as $\exp(-\mu|x|)$, $|F|$ falls fast and we have a short distance contribution of F which is negative in the Landau gauge.

B. Analytic continuation of $S_F(p)$

Here we show the analytic form of the quenched fermion propagator without mass changing effects in Minkowski space by using the formulae

$$\operatorname{arctanh}(z) = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) = -i \operatorname{arctan}(iz), (0 \leq z^2 \leq 1) \quad (77)$$

$$\operatorname{arctanh}(z) = \frac{1}{2} \ln\left(\frac{1+z}{z-1}\right) \pm \frac{i\pi}{2}, \quad (78)$$

$$\operatorname{arccoth}(z) = \frac{1}{2} \ln\left(\frac{1+z}{z-1}\right), (1 \leq z^2). \quad (79)$$

Principal part of the propagator in Minkowski space is continued to

$$\begin{aligned} S_F(p) &= \frac{(\gamma \cdot p + m)c^D \Gamma(D+1) \sinh((D+1)\operatorname{arctanh}(\sqrt{p^2}/m))}{\sqrt{p^2}(m^2 - p^2)^{(D+1)/2}}, (|\sqrt{p^2}/m| \leq 1) \\ &= \frac{(\gamma \cdot p + m)c^D \Gamma(D+1) \sinh((D+1)\operatorname{arccoth}(\sqrt{p^2}/m))}{\sqrt{p^2}(p^2 - m^2)^{(D+1)/2}}, (|\sqrt{p^2}/m| \geq 1). \end{aligned} \quad (80)$$

The spectral function is a discontinuity in the upper half-plane of $z^2 = p^2/m^2 \geq 1$

$$\pi\rho(z) = -\frac{c^D}{2} \operatorname{Im} \frac{\Gamma(D+1)}{m|z|(m^2 - m^2 z^2)^{(D+1)/2}} \left[\left(\frac{1+z}{1-z}\right)^{(D+1)/2} - \left(\frac{1-z}{1+z}\right)^{(D+1)/2} \right]. \quad (81)$$

For $z^2 \geq 1$

$$\left(\frac{1}{1-z^2}\right)^{(D+1)/2} = \left|\frac{1}{1-z^2}\right|^{D+1} \exp(-(D+1)\pi i/2), \quad (82)$$

$$\left(\frac{1+z}{1-z}\right)^{(D+1)/2} = \left|\frac{1+z}{1-z}\right|^{(D+1)/2} \exp(-(D+1)\pi i/2), z \geq 1 \quad (83)$$

$$\left(\frac{1-z}{1+z}\right)^{(D+1)/2} = \left|\frac{1-z}{1+z}\right|^{(D+1)/2} \exp(-(D+1)\pi i/2), z \leq -1 \quad (84)$$

we have

$$\pi\rho(z) = \frac{c^D \Gamma(D+1) \sin((D+1)\pi)}{2m|z|m^{D+1}} \left[\left(\frac{1}{z-1}\right)^{D+1} \theta(z-1) - \left|\frac{1}{z+1}\right|^{D+1} \theta(-(z+1)) \right]. \quad (85)$$

The vanishment of the renormalization constant

$$Z_2^{-1} = \int z dz \rho(z) = 0 \quad (86)$$

which is evaluated as the high energy behaviour of $S_F(p)$ (49) is seen by

$$I = \int_{1+\epsilon}^{\infty} dz \left(\frac{1}{z-1} \right)^{D+1} + \int_{-\infty}^{-1-\epsilon} dz \left| \frac{1}{z+1} \right|^{D+1} = 0, \quad (87)$$

where $\epsilon = \mu/m$.